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ANALYTICAL EXPRESSIONS FOR STEADY MHD BOUNDARY LAYER FLOW OVER A STRETCHING SURFACE

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Abstract:

This research paper is to study the analytical expressions for boundary layer flow with diffusion and chemical reaction past a stretching. The governing partial differential equations are transformed into the ordinary differential equations. The approximate analytical solutions of the dimensionless stream function and the dimensionless concentration are derived by using the modified Homotopy analysis method. The features of various physical parameters have been discussed graphically on flow of the dimensionless concentrations. We can also compare our analytical solution of the dimensionless stream function with the exact solution and a satisfied agreement is noted. The Homotopy analysis method contains the convergence control parameter h, so it can be easily extended to solve the other non-linear initial and boundary value problems in other mathematical sciences.

Key Words: Boundary Layer Flow, Newtonian Fluid, Non-Linear Differential Equations & Modified Homotopy Analysis Method.

1. Introduction:

Analytical expressions for steady MHD boundary layer flow over a stretching surface are relevant to many engineering problems such as paper production preparing plastic and metal sheets. The chemical reaction effects were studied by many researchers on several physical aspects. Sakiadis [9-10] studied the laminar boundary layer flow caused by a rigid surface moving in its own plane. Crane [5] extended the works of Sakiadis and considers the flow transfer analysis in Newtonian boundary layer floes past a stretching sheet was studied by Gupta et.al [7]. Chakrabarti et.al [3] analyzed the MHD flow of Newtonian fluid initially at rest over a stretching sheet at different values of parameter of parameter with uniform temperature. Cornell [4] studied the motion and mass transfer for two classes of viscoelastic fluid over a porous stretching sheet immersed in a porous medium. A special form of Lie group of transformation is known as a scaling group of transformation is used to find out the full set of symmetric of the flow problem. Afify [8] explicated the MHD free convective flow of viscous incompressible fluid and mass transfer over a stretching sheet with chemical reaction. Crane [5] was the first to consider and examine the boundary layer flow of a viscous fluid over a non -linearly stretching in recent decades non –Newtonian fluids more important than Newtonian fluids Gupta and Gupta studied heat and mass transfer over a continuous stretching surface. Kandasamy et.al [6] studied the effects of temperature dependent fluid viscosity and chemical reaction on heat and mass transfer with variable stream function.

In the present investigation, we derive the analytical expressions for steady MHD boundary layer flow over a stretching surface by usingthe Modified Homotopy analysis method. Comparison of the analytical solution and exact solution of dimensionless stream function are also reported in table format.

2. Mathematical Formulation of the Problem

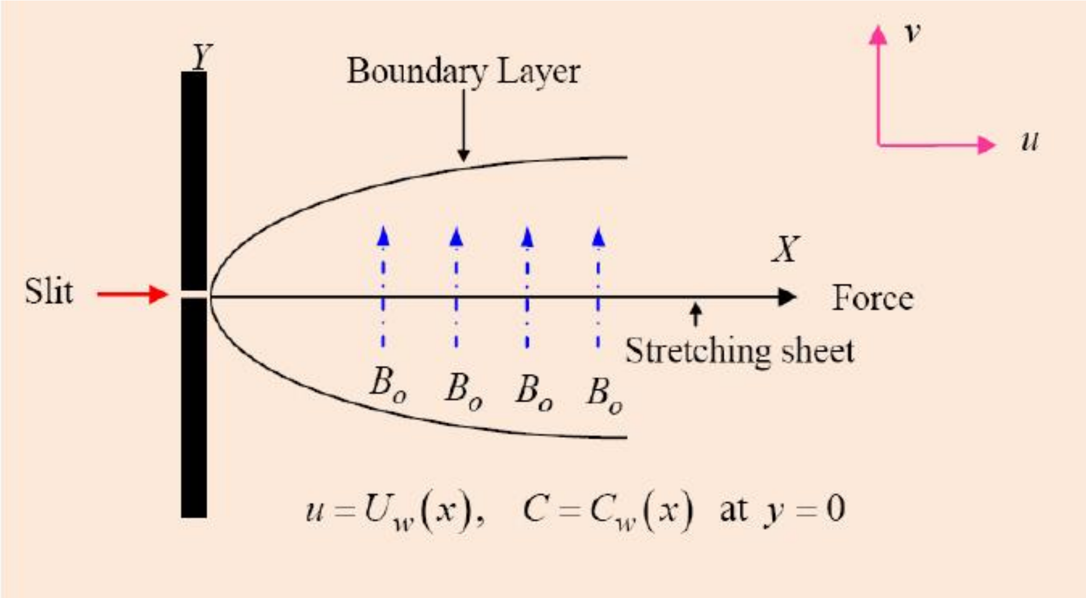

Fig.1: Schematic of two-dimensional stretching sheet

The physical situation considered for the investigation is that of a steady state, laminar boundary layer flow of an electrically conducting incompressible viscous fluid in the presence of a transverse magnetic field B_0 due to a stretching horizontal sheet as shown in fig.1 the flow is generated by the action of two equal and opposite forces along the X -axis and the sheet is stretched with a velocity that is proportional to the distance from the slit. Let the concentration at the stretching sheet is $C_w(x) = cx$, c is constant .the stretching sheet assumed velocity of the form $U_w(x) = bx$ where b is the stretching constant and x is the distance from the slit .it is also assumed that the magnetic Reynolds number Re_m is very small ; i .e. $\mathrm{Re}\,\mu_0 \sigma b L << 1$ where μ_0 is the magnetic permeability , L is the permeability , L is the reference length and σ is the electric conductivity .We neglected the induced magnetic field, which is small in comparison with the applied magnetic field. Applying boundary layer approximation [2] governing the continuity, momentum and concentration equation may be written in usual notation as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} - Mu \qquad (2)$$

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\frac{\partial^2 c}{\partial y^2} - kc \qquad (3)$$

In the above eqns., u and v are the velocity components along the x and y axes respectively, k is the kinematic velocity ρ is the fluid density, C is the concentration, D is diffusion coefficient, K denotes the reaction rate constant

The relevant boundary conditions applicable to the flow are:

$\mathrm{Re} = \frac{U_0 L}{v}$ is represents the Reynolds number,

$$\left.\begin{aligned} &u(x,0)=U_w(x)[=bx], v(x,0)=0 \\ &C(x,0)=C_w(x)[=cx] \\ &v(x,y)=C(x,y)=0 \quad as \quad y\to\infty \end{aligned}\right\} \tag{4}$$

We introduce following dimensionless quantity:

$$x^*=\frac{x}{L}, \quad y^*=\sqrt{Re}\frac{y}{L}, \quad u^*=\frac{u}{U_{0,}} v^*=\sqrt{Re}\frac{v}{U_0}, \quad C^*=\frac{C}{C_0} \tag{5}$$

Where $U_0=bL$ represent reference moving speed,

$C_0=cL$ is represents the reference concentration.

Dropping the asterisks the boundary layer becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{6}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=v\frac{\partial^2 u}{\partial y^2}-Mu \tag{7}$$

$$u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}=\frac{1}{Sc}\frac{\partial^2 c}{\partial y^2}-\delta C \tag{8}$$

In which $Sc=\frac{V}{D}$ represents Schmidt number,

$\delta=\frac{KL}{U_0}$ represents reaction rate parameter,

$M=\frac{\sigma B_0^{\ 2}}{\rho L}$ represents magnetic field strength parameter.

The boundary conditions become

$$\left.\begin{aligned} &u(x,0)=x \\ &v(x,0)=0 \\ &C(x,y)=x \\ &u(x,y)=C(x,y)=0 \ as \ y\to\infty \end{aligned}\right\} \tag{9}$$

By introducing the dimensionless stream function $u(x,y)$ such that $u=\frac{\partial\psi}{\partial y}$ and $v=\frac{\partial\psi}{\partial x}$

satisfies the eqn.(6) identically. Then the eqns. (7) and (8) becomes

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y^2}=\frac{\partial^3\psi}{\partial y^3}-M\frac{\partial\psi}{\partial y} \tag{10}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial C}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial C}{\partial y}=\frac{1}{SC}\frac{\partial^2 C}{\partial y^2}-\delta C \tag{11}$$

The corresponding boundary conditions are as follows:

$$\left.\begin{aligned}&\frac{\partial\psi}{\partial y}(x,0)=x,\sqrt{2\frac{\partial\psi}{\partial x}}(x,0)=0,\\&C(x,0)=x\\&\frac{\partial\psi}{\partial y}(x,y)=C(x,y)=0\ as\ y\to\infty\end{aligned}\right\}\qquad(12)$$

Subgroup of transformation, through which one will reduce two independent variable by one and the system of non- linear partial differential eqns. (9) to (11) will transform to the system of ordinary differential equation

$$C_G : T_a(Q)=\aleph^Q(a)s+\Re^Q(a)=\overline{Q} \qquad (13)$$

Where Q stands for x, y,ψ,ϕ whereas $\aleph's$ and $\Re's$ are at least differentiable in the real argument a

To transform the differential transformation of the derivatives ψ are obtained

From G_C via chain rule operations:

$$\left.\begin{aligned}&\overline{s}_{\overline{i}}=\left(\frac{\aleph^Q}{\aleph^i}\right)Q_i\\&\overline{Q}_{\overline{i}\,\overline{j}}=\left(\frac{\aleph^Q}{\aleph^i\aleph^j}\right)Q_{ij}\end{aligned}\right\}Q=\psi,C;\quad i,j=x,y \qquad (14)$$

The eqns. (10) and (11) are said to be invariantly transformed for some function $\xi_1(a)$ and $\xi_2(a)$ whenever

$$\frac{\partial\overline{\psi}}{\partial\overline{y}}\frac{\partial^2\overline{\psi}}{\partial\overline{y}\partial\overline{x}}-\frac{\partial\overline{\psi}}{\partial\overline{x}}\frac{\partial^2\overline{\psi}}{\partial\overline{y}^2}-\frac{\partial^3\psi}{\partial\overline{y}^3}+M\frac{\partial\overline{\psi}}{\partial\overline{y}}=\xi_1(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^3\psi}{\partial y^3}+M\frac{\partial\psi}{\partial y}\right] \qquad (15)$$

$$\frac{\partial\overline{\psi}}{\partial\overline{y}}\frac{\partial^2\overline{\psi}}{\partial\overline{y}\partial\overline{x}}-\frac{\partial\overline{\psi}}{\partial\overline{x}}\frac{\partial^2\overline{\psi}}{\partial\overline{y}^2}-\frac{\partial^3\psi}{\partial\overline{y}^3}+M\frac{\partial\overline{\psi}}{\partial\overline{y}}=\xi_1(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^3\psi}{\partial y^3}+M\frac{\partial\psi}{\partial y}\right] \qquad (16)$$

Substituting the value from the eqns. (13) and (14) in above system of equation,

$$\begin{aligned}&\frac{(\aleph^\psi)^2}{\aleph^x(\aleph^y)^2}\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right]-\frac{\aleph^\psi}{(\aleph^y)^3}\frac{\partial^3\psi}{\partial y^3}+M\frac{\aleph^\psi}{\aleph^y}\frac{\partial\psi}{\partial y}\\&=\zeta_1(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^3\psi}{\partial y^3}+\frac{\partial\psi}{\partial y}\right]\end{aligned} \qquad (17)$$

$$\begin{aligned}&\frac{\aleph^\psi\aleph^C}{\aleph^x\aleph^y}\left[\frac{\partial\psi}{\partial y}\frac{\partial C}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial C}{\partial y}\right]-\frac{1}{Sc}\frac{\aleph^C}{(\aleph^y)^2}\frac{\partial^2C}{\partial y^2}\delta(\aleph^C C+\Re^C)\\&=\zeta_2(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial C}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial C}{\partial y}-\frac{1}{Sc}\frac{\partial^2C}{\partial y^2}+\delta C\right]\end{aligned} \qquad (18)$$

The invariance of the eqns. (16) and (17) together with boundary conditions, implies that

$$\frac{(\aleph^\psi)^2}{\aleph^x(\aleph^y)^2}=\frac{\aleph^\psi}{(\aleph^y)^3}\frac{\aleph^\psi}{\aleph^y}=\zeta_1(a) \qquad (19)$$

$$\frac{\aleph^{\psi}\aleph^{C}}{\aleph^{x}\aleph^{y}} = \frac{\aleph^{C}}{\left(\aleph^{y}\right)^{2}} = \aleph^{C} = \zeta_2(a) \qquad (20)$$

These yields,

$$\aleph^{x} = \aleph^{C} = \aleph^{\psi}, \quad \Re^{x} = \Re^{C} = \Re^{\psi} = \Re^{y} = 0 \qquad (21)$$

The one –parameter group G that transform invariantly the differential eqns.(10) and (11)with
the auxiliary conditions (9) is,

$$G: \begin{cases} G_H: \begin{cases} \bar{x} = \aleph^{x} x \\ \bar{y} = y \end{cases} \\ \bar{\psi} = \aleph^{x}\psi \\ \bar{C} = \aleph^{x} C \end{cases} \qquad (22)$$

$$\eta = y, \quad \psi = xf(\eta), \quad C = xg(\eta) \qquad (23)$$

Substituting the values of partial derivatives from (23) into eqn. (10) and (11), we get the following ordinary differential equations [1]:

$$f''' - f'^{2} + ff' - Mf' = 0 \qquad (24)$$

$$\frac{1}{Sc} g'' + fg' - \delta g = 0 \qquad (25)$$

The corresponding boundary conditions are as follows:

$$f(0) = 0, f'(0) = 1, f'(\infty) = 0 \qquad (26)$$

$$g(0) = 1, g(\infty) = 1 \qquad (27)$$

3. Solution of the Problem Using the Modified Homotopy Analysis Method:

HAM is a non-perturbative analytical method for obtaining series solutions to nonlinear equations and has been successfully applied to numerous problems in science and engineering [11-31]. In comparison with other perturbative and non-perturbative analytical methods, HAM offers the ability to adjust and control the convergence of a solution via the so-called convergence-control parameter. Because of this, HAM has proved to be the most effective method for obtaining analytical solutions to highly non-linear differential equations. Previous applications of HAM have mainly focused on non-linear differential equations in which the non-linearity is a polynomial in terms of the unknown function and its derivatives. As seen in (1), the non-linearity present in electro hydrodynamic flow takes the form of a rational function, and thus, poses a greater challenge with respect to finding approximate solutions analytically. Our results show that even in this case, HAM yields excellent results.

Liao [11-19] proposed a powerful analytical method for non-linear problems, namely the Homotopy analysis method. This method provides an analytical solution in terms of an infinite power series. However, there is a practical need to evaluate this solution and to obtain numerical values from the infinite power series. In order to investigate the accuracy of the Homotopy analysis method (HAM) solution with a finite number of terms, the system of differential equations were solved. The Homotopy analysis method is a good technique comparing to another perturbation method. The Homotopy analysis method contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of solution series. The approximate analytic solution of the eqns. (24)-(27) by using the Modified HAM is as follows:

$$f=\frac{1-e^{-M\eta}}{M}-\frac{\left(1+\frac{1}{M}\right)}{8M^3}e^{-2M\eta}-\frac{\left(M-\frac{1}{M}\right)}{M^3}e^{-M\eta}-\left(\frac{\left(\left(1+\frac{1}{M}\right)+4\left(M-\frac{1}{M}\right)\right)}{4M^2}\right)\eta+\frac{\left(\left(1+\frac{1}{M}\right)+8\left(M-\frac{1}{M}\right)\right)}{8M^3} \quad (28)$$

$$g=e^{-Sc\eta}+\frac{e^{-Sc\eta}}{M}-\frac{Sc^2e^{-Sc\eta-M\eta}}{M(-Sc-M)^2}+\delta\frac{e^{-Sc\eta}}{Sc}-\frac{1}{M}+\frac{Sc^2}{M(-Sc-M)^2}-\delta\frac{1}{Sc} \quad (29)$$

The exact solution [1] of the above eqn.(24) subject to the boundary condition eqn. (26) of the form:

$$f(\eta)=\frac{1-e^{-m\eta}}{m},\ where\ m=\sqrt{1+M} \quad (30)$$

The mathematical expression for local skin-friction coefficient C_{fx} is given by

$$\frac{1}{2}\sqrt{\mathrm{Re}_x}C_{fx}=-f''(0)=\sqrt{1+M} \quad (31)$$

where $\sqrt{\mathrm{Re}_x}$ represents the local Reynolds number

4. Results and Discussion:

Figure (2) represents dimensionless local-skin friction coefficient C_{fx} versus Dimensionless horizontal coefficient (η).From this Fig., it is evident that when the dimensionless parameter increases the corresponding local skin-friction coefficient also increases. Fig.(3)-(5) shows the dimensionless concentration $g(\eta)$ versus the dimensionless horizontal coordinate (η).From this fig. its observed that when Schmidt number Sc increases the corresponding dimensionless concentration decreases in some fixed value of the other dimensionless parameter M and δ.

Figure (4) it is noted that when dimensionless parameter M increases the corresponding dimensionless concentration $g(\eta)$ also increases very small in some fixed value of the other dimensionless parameter Sc and δ. Fig.(5) it is noted that when the dimensionless parameter δ increases the corresponding the dimensionless concentration decrease $g(\eta)$ in some fixed value of other dimensionless parameter Sc and M. Table.1 represents the comparing of our analytical solution and exact solution for the stream function $f(\eta)$.

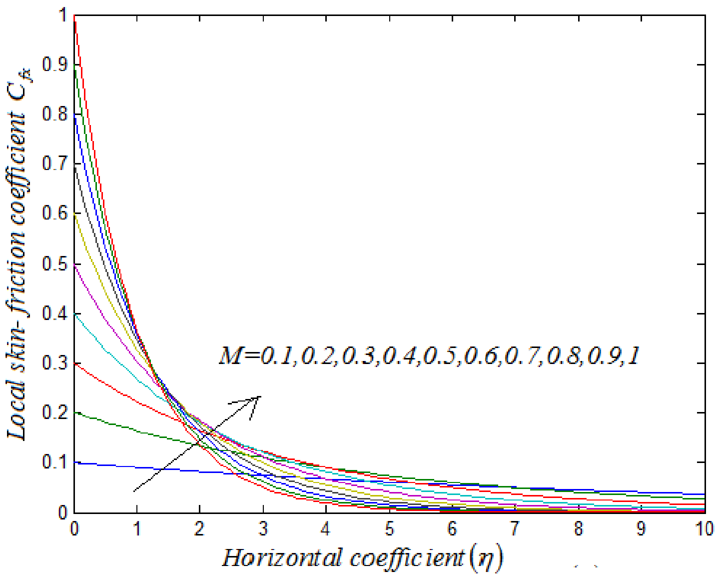

Fig.2: Dimensional horizontal coordinate versus local skin friction C_{fx} .The curves are plotted for various value of magnetic field strength M using the eqn.(B.17).

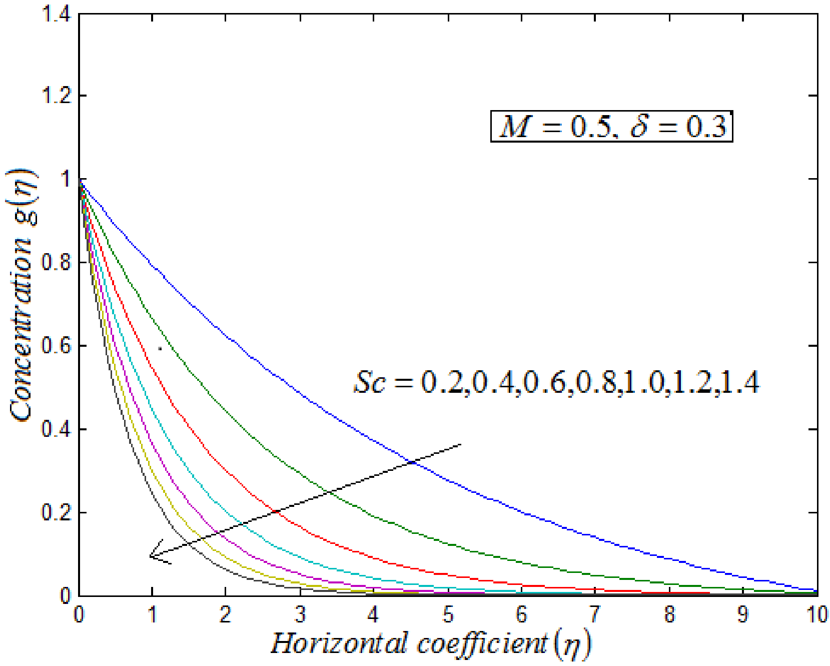

Fig.3: Dimensionless concentration $g(\eta)$ versus Dimensional horizontal coordinate η .The curves are plotted for various values of the dimensionless parameter Sc and fixed value of the other parameter M, δ using the eqn. (29) with $h = -0.0072$

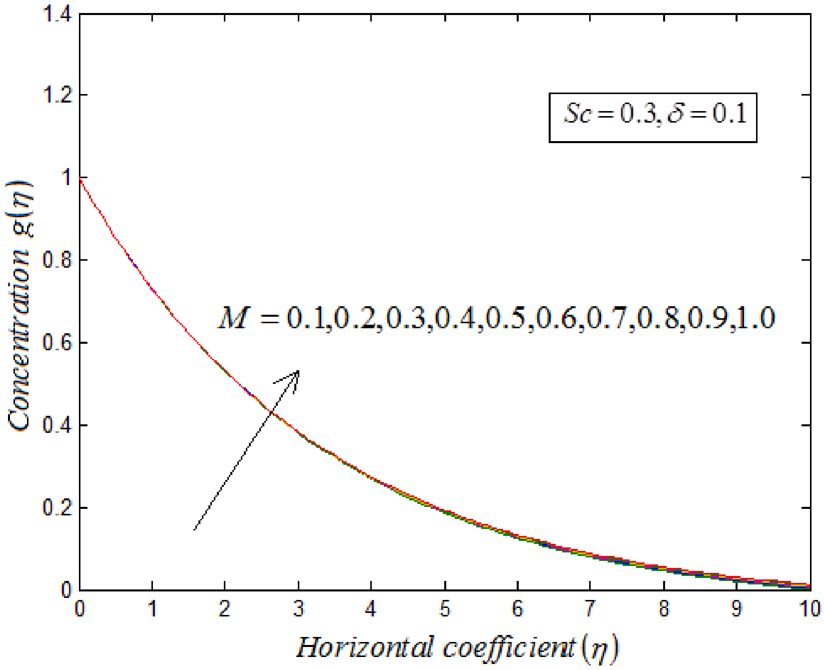

Fig. 4: Dimensionless concentration $g(\eta)$ versus Dimensional horizontal coordinate η .The curves are plotted for various values of the dimensionless parameter M and fixed value the other parameter Sc, δ using the eqn. (29) with $h = -0.02179$

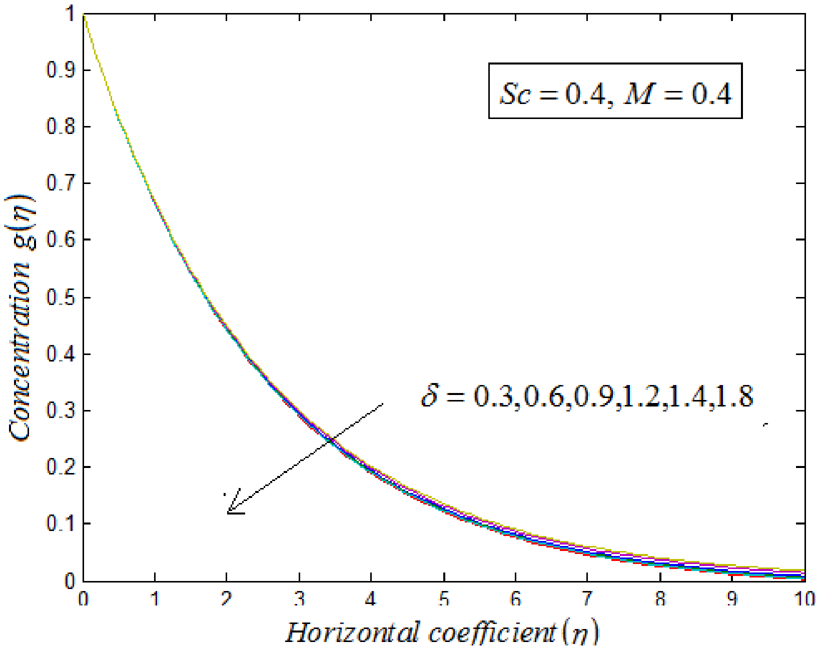

Fig.5: Dimensionless concentration $g(\eta)$ versus Dimensional horizontal coordinate η. the curves are plotted for various values of the dimensionless parameter δ and fixed value of the other parameter M, Sc using the eqn. (29) with $h =$ -0.0026

Table:1 Comparison of the exact solution of $f(\eta)$ eqn. (30)and Modified HAM solution of $f(\eta)$

	Exact solution $f(\eta)$	HAM solution $f(\eta)$ $h = -(0.4033)$	Exact solution $f(\eta)$	HAM solution $f(\eta)$ $h = -(0.4569)$	Exact solution $f(\eta)$	HAM solution $f(\eta)$ $h = -(0.7396)$
η	M =0.5	M =0.5	M =0.4	M =0.4	M =1	M =1
0	0	0	0	0	0	0
1	-0.5765	-0.5765	-0.5862	-0.5862	-0.5351	-0.5351
2	-0.7460	-0.7460	-0.7658	-0.7658	-0.6653	-0.6653
3	-0.7957	-0.7956	-0.8208	-0.8208	-0.6969	-0.6969
4	-0.8104	-0.8103	-0.8377	-0.8377	-0.7046	-0.7046
5	-0.8147	-0.8147	-0.8428	-0.8428	-0.7065	-0.7065
6	-0.8159	-0.8158	-0.8444	-0.8444	-0.7069	-0.7069
7	-0.8163	-0.8163	-0.8449	-0.8449	-0.7070	-0.7070
8	-0.8164	-0.8163	-0.8450	-0.8450	-0.7070	-0.7070
9	-0.8164	-0.8164	-0.8451	-0.8451	-0.7171	-0.7171

10	-0.8164	-0.8164	-0.8451	-0.8451	-0.7171	-0.7171

5. Conclusion:

The present paper is modeled the analytical expressions for steady MHD boundary layer flow over a stretching surface. The approximate analytical expressions of the dimensionless stream function and the dimensionless concentration are derived by using Modified Homotopy analysis method. Our analytical solution of the stream function is compared with the exact solution. The analytical expression for the dimensionless local skin-friction coefficient is also derived analytically and graphically.

6. Acknowledgement:

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Appendix A

Basic Concept of the Homotopy Analysis Method [11-31]

Consider the following differential equation:

$$N[u(t)] = 0 \qquad (A.1)$$

Where N is a nonlinear operator, t denote an independent variable, $u(t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional Homotopy method, Liao (2012) constructed the so-called zero-order deformation equation as:

$$(1-p)L[\varphi(t;p)-u_0(t)] = phH(t)N[\varphi(t;p)] \quad \text{(A.2)}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L an auxiliary linear operator, $u_0\ (t)$ is an initial guess of $u(t)$, is an unknown function. It is important to note that one has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\varphi(t;0) = u_0(t) \text{ and } \varphi(t;1) = u(t) \quad \text{(A.3)}$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $u_0(t)$ to the solution $u\ (t)$. Expanding $\varphi(t;p)$ in Taylor series with respect to p, we have: $\varphi(t;p) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)p^m$ (A.4)

Where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \varphi(t;p)}{\partial p^m}\Big|_{p=0} \quad \text{(A.5)}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (A.4) converges at p =1 then we have:

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t). \quad \text{(A.6)}$$

Differentiating (A.2) for m times with respect to the embedding parameter p, and the n setting p = 0 and finally dividing them by m, we will have the so-called *math*-order deformation equation as:

$$L[u_m - \chi_m u_{m-1}] = hH(t)\Re_m(\vec{u}_{m-1}) \quad \text{(A.7)}$$

Where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\varphi(t;p)]}{\partial p^{m-1}} \quad \text{(A.8)}$$

and $\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$ (A.9)

Applying L^{-1} on both side of equation (A7), we get

$$u_m(t) = \chi_m u_{m-1}(t) + hL^{-1}[H(t)\Re_m(\vec{u}_{m-1})] \quad \text{(A10)}$$

In this way, it is easily to obtain u_m for $m \ge 1$, at M^{th} order, we have

$$u(t) = \sum_{m=0}^{M} u_m(t) \quad \text{(A.11)}$$

When $M \to +\infty$, we get an accurate approximation of the original equation (A.1). For the convergence of the above method we refer the reader to Liao [20]. If equation (A.1) admits unique solution, then this method will produce the unique solution.

Appendix B

Approximate Analytical Expression of the Non - Linear Differential Eqns. (24)-(27) Using the Modified Homotopy Analysis Method:

In this appendix, we derive the analytical expressions for $f(\eta)$and $g(\eta)$using the modified HAM. The eqns.(24) and (25)can be write in the following form

$$f'''-f'^2+ff'-Mf'=0 \qquad \text{(B.1)}$$

$$\frac{1}{Sc}g''+fg'-\delta g=0 \qquad \text{(B.2)}$$

We construct the Homotopy are as follows

$$(1-p)\frac{d^3f}{d\eta^3}=hp\left[\frac{d^3f}{d\eta^3}-\left(\frac{df}{d\eta}\right)^2+f\frac{df}{d\eta}-M\frac{df}{d\eta}\right] \qquad \text{(B.3)}$$

$$(1-p)\frac{d^2g}{d\eta^2}=hp\left[\frac{d^2g}{d\eta^2}-Sc\left[\delta g-f\frac{dg}{d\eta}\right]\right] \qquad \text{(B.4)}$$

The approximate analytical solution of the eqn.(B.3) and (B.4) are as follows:

$$f=f_0+pf_1+..... \qquad \text{(B.5)}$$

$$g=g_0+pg_1+..... \qquad \text{(B.6)}$$

Substituting the eqn.(B.5) in to the eqn. (B.3) and(B.6) in to the eqn.(B.4) we get

$$(1-p)\frac{d^3(f_0+pf_1+.....)}{d\eta^3}=hp\left[\frac{d^3(f_0+pf_1+.....)}{d\eta^3}-\left(\frac{d(f_0+pf_1+...)}{d\eta}\right)^2+(f_0+pf_1+...)\frac{d(f_0+pf_{1+....})}{d\eta}-M\frac{d(f_0+pf_1+...)}{d\eta}\right] \qquad \text{(B.7)}$$

$$(1-p)\frac{d^2(g_0+pg_1+.....)}{d\eta^2}=hp\left[\frac{d^2(g_0+pg_1+.....)}{d\eta^2}-Sc\left[\delta(g_0+pg_1+.....)-(f_0+pf_1+....)\frac{d(g_0+pg_1+.....)}{d\eta}\right]\right] \qquad \text{(B.8)}$$

Comparing the coefficients of like powers of p the eqn.(B.7) and (B.8) we get following eqns:

$$p^0:\frac{d^3f_0}{d\eta^3}=0 \qquad \text{(B.9)}$$

$$p^0:\frac{d^2g_0}{d\eta^2}=0 \qquad \text{(B.10)}$$

$$p^1=\frac{d^3f_1}{d\eta^3}-\frac{d^3f_0}{d\eta^3}=h\left[\frac{d^3f_0}{d\eta^3}+\left(\frac{d\ \ f_0}{d\eta}\right)^2+f_0\frac{df_0}{d\eta}-M\frac{df_0}{d\eta}\right] \qquad \text{(B.11)}$$

$$p^1=\frac{d^2g_1}{d\eta^2}-\frac{d^2g_0}{d\eta^2}=h\left[\frac{d^2g_0}{d\eta^2}-Sc\left[\delta g_0-f_0\frac{dg_0}{d\eta}\right]\right] \qquad \text{(B.12)}$$

We choose the initial guesses in the following form which satisfies the eqn. (B.13)

$$f_0 = \frac{1-e^{-M\eta}}{M} \tag{B.13}$$

$$g_0 = e^{-Sc\eta} \tag{B.14}$$

The initial approximations are as follows:

$$f_0(0)=0,\ f_0'(0)=1,\ f'_0(\infty)=0 \text{ and } g_0(0)=1,\ g_0(\infty)=1 \tag{B.15}$$

$$f_i(0)=0,\ f_i'(0)=0,\ f'_i(\infty)=0 \text{ and } g_i(0)=0,\ g_i(\infty)=0 \tag{B.16}$$

By solving the eqn. (B.16) using the boundary conditions (B.15).We get the expression for local skin-friction coefficient C_{fx} is

$$-f_0'''(\eta)=M \tag{B.17}$$

By solving the eqns.(B.11)and (B.12)using the eqn.(B.16)we can obtain the following results:

$$f_1 = -\frac{\left(1+\frac{1}{M}\right)}{8M^3}e^{-2M\eta} - \frac{\left(M-\frac{1}{M}\right)}{M^3}e^{-M\eta} - \left(\frac{\left(\left(1+\frac{1}{M}\right)+4\left(M-\frac{1}{M}\right)\right)}{4M^2}\right)\eta + \frac{\left(\left(1+\frac{1}{M}\right)+8\left(M-\frac{1}{M}\right)\right)}{8M^3} \tag{B.18}$$

$$g_1 = \frac{e^{-Sc\eta}}{M} - \frac{Sc^2 e^{-Sc\eta-M\eta}}{M(-Sc-M)^2} + \delta\frac{e^{-Sc\eta}}{Sc} - \frac{1}{M} + \frac{Sc^2}{M(-Sc-M)^2} - \delta\frac{1}{Sc} \tag{B.19}$$

According to the Modified HAM, we conclude that

$$f = \lim_{p\to\infty} f(\eta) = f_0 + f_1 \tag{B.20}$$

$$g = \lim_{p\to\infty} g(\eta) = g_0 + g_1 \tag{B.21}$$

After putting eqns. (B.13)and(B.18)into an eqns.(B.20) and the eqns.(B.14)and(B.19) into the eqns.(B.21).we obtain the solution in the text as given in the eqns.(27) and (28).

Appendix C:

Nomenclature:

Symbol	Meaning
η	Dimensional horizontal coordinate
Sc	Schmidt number
δ	Reaction rate parameter
M	Magnetic field strength parameter
Re	Reynolds number
C_0	Concentration
U_0	Moving speed
ψ	Stream function
ρ	Fluid density

Γ	Scaling group transformation
k	Reaction rate constant
a	Stretching constant
g	Dimensionless concentration
C_{fx}	Local Skin –friction coefficient
u, v	velocity components