



## **A BRIEF STUDY ON SIMULATION OF A QUEUEING SYSTEM**

**A. Dinesh Kumar\*, M. Vasuki\*\* & S. Kumar\***

\* Assistant Professor, Department of Mathematics, Dhanalakshmi Srinivasan Engineering College, Perambalur, Tamilnadu

\*\* Assistant Professor, Department of Mathematics, Srinivasan College of Arts and Science, Perambalur, Tamilnadu

### **Introduction:**

Simulation is a numerical technique for conducting experiments that involve certain types of mathematical and logical relationships necessary to describe the behavior and structure of a complex real world system over extended period of time. Simulation is the process of designing a model of a real system and conducting experiments with this model for the purpose of understanding the behavior (within the limits imposed by a criterion or set of criteria) for the operation of the system.

### **Why Simulation?**

Using simulation, an analyst can introduce the constants and variables related to the problem, set-up the possible courses of action and establish criteria which act as measures of effectiveness. The major reasons for applying simulation technique to O.R problems may be listed as below:

- ✓ It is an appropriate tool to use in solving a problem when experimenting on the real system.
  - Would be disruptive.
  - Would be too expensive.
  - Does not permit replication events.
  - Does not permit control over key variables.
- ✓ It is a desirable tool for solving a business problem when a mathematical model.
  - Is too complex to solve.
  - Is beyond the capacity of available personnel.
  - Is not detailed enough to provide information on all important decision variables.
- ✓ The major reasons for adopting simulation in place of other mathematical techniques are:
  - It may be the only method available, because it is difficult to observe the actual reality.
  - Without appropriate assumption, it is impossible to develop a mathematical solution.
  - It may be too expensive to actually observe the system.
  - There may not be sufficient time to allow the system to operate for a very long time.
- ✓ It provides a trial-and-error movement towards the optimal solution. The decision-maker selects an alternative, experiences the effect of the selection and then improves the selection. In this way, the selection is adjusted until it approximates the optimal solution.

### **Methodology of Simulation:**

The methodology developed for simulation process consists of the following seven steps:

- ✓ Identify and clearly define the problem.
- ✓ List the statement of objectives of the problem.

- ✓ Formulate the variables that influence the situation and an exact or probabilistic description of their possible values or states.
- ✓ Obtain a consistent set of values for the variables. (ie) a sample of what could happen. In the case of deterministic variables, this is simple and in the case of probabilistic variables, random sampling technique may be used.
- ✓ Use the sample obtained in step 2 to calculate the value of the decision criterion, by actually following the relationships among the variables for each of the alternative decisions.
- ✓ Repeat steps 2 and 3 until a sufficient number of samples are available.
- ✓ Tabulate the various values of the decision criterion and choose the best policy.

#### **Simulation Models:**

The simulation models can be classified into the following four categories.

- ✓ **Simulation of Deterministic Models:** In the case of these models, the input and output variables are not permitted to be random variables and models are described by exact functional relationships.
- ✓ **Simulation of Probabilistic Models:** In such cases, method of random sampling is used. The technique used for solving these models is termed as "Monte-Carlo-Technique".
- ✓ **Simulation of Static Models:** These models do not take variable time into consideration.
- ✓ **Simulation of Dynamic Models:** These models deal with time-varying interaction.

#### **The Advantages of Simulation:**

- ✓ The study of very complicated system or sub-system can be done with the help of simulation. Simulation has been described as "what to do when all else fails".
- ✓ By using simulation, we can investigate the consequences for a system of possible changes in parameters in terms of the model.
- ✓ The knowledge of a system obtained in designing and conducting the simulation is very valuable.
- ✓ It enables us to assess the possible risks involved in a new policy before actually implementing it.
- ✓ The simulation of complicated systems helps us to locate which variables have the important influences on system performance.
- ✓ Simulation methods are easier to apply than pure analytical methods.
- ✓ It is a teaching aid, (for example) in business games, case studies, etc.
- ✓ This can be used to experiment unfamiliar systems to prepare routine and extreme eventualities.

#### **The Limitations of Simulation:**

- ✓ Simulation generates a way of evaluating solutions but it does not generate the solution techniques.
- ✓ Sometimes simulation models are expensive and take a long time to develop it. For example, a corporate planning model may take a long time to develop and prove expensive also.
- ✓ The simulation model does not produce answers by itself. The user has to provide all the constraints for the solutions which he wants to examine.
- ✓ Not all situations can be evaluated using simulation. Only situations involving uncertainly are considered.
- ✓ It is the trial-and-error approaches that produce different solutions in repeated runs. This means it does not generate optimal solutions to problems.

- ✓ Simulation is a time-consuming exercise.

#### **Random Number Generation:**

The following seven steps explain this method:

- ✓ Collect the data related to the current problem.
- ✓ Construct a frequency distribution with these data.
- ✓ Construct the relative frequency distribution.
- ✓ Assign a coding system that relates the identified events to generated random numbers.
- ✓ Select a suitable method for obtaining the required random numbers.
- ✓ Match the random number to the assigned events and tabulate the results.
- ✓ Repeat step 6 until desired number of simulation runs has been generated.

#### **Illustration:**

Kodak Photography studios use an expensive of developing fluid when printing special colour portraits. Since the developing fluid cannot be stored for long periods, it is important to keep on hand only as much as is needed to fill anticipated demand. In the past few months, however, demand for the product has been fluctuating. The owner has decided to simulate the demand for this service. A study of Kodak Photography's appointment book resulted in the following frequency distribution.

<i>Daily Demand</i>	0	1	2	3	4	5
<i>Number of Days</i>	10	20	40	20	6	4

The data was taken for a 100 day's period during which no more than 5 special prints were requested on any given day.

35 92 68 03 51 05 72 84 98 34

Using the above given data to generate a 10 day sequence of demand values.

#### **Solution:**

Collect the data relevant to the problem and construct the frequency distribution and also construct the corresponding relative frequency distribution.

#### **Relative Frequency Distribution:**

<i>Daily Demand</i>	<i>Relative Frequency</i>	<i>Probability</i>
0	10/100	0.10
1	20/100	0.20
2	40/100	0.40
3	20/100	0.20
4	6/100	0.06
5	4/100	0.04

#### **Random Number Assignment:**

<i>Daily Demand</i>	<i>Probability</i>	<i>Random Number Assignment</i>	<i>Number Assigned</i>
0	0.10	00 – 09	10
1	0.20	10 – 29	20
2	0.40	30 – 69	40
3	0.20	70 – 89	20
4	0.06	90 – 95	6
5	0.04	96 – 99	4

Since the owner wants a 10 days simulation by using the generated random number given in the problem. We find the generated demand in the following table.

#### **Randomly Generated Data:**

<i>Day Number</i>	<i>Generated Random Demand</i>	<i>Generated Demand</i>
1	35	2
2	92	4
3	68	2
4	03	0

5	51	2
6	05	0
7	72	3
8	84	3
9	98	5
10	34	2

**Total**

**23**

Total generated demand = 23 units

Total number of days in the simulation = 10

$$\therefore \text{Average daily demand} = \frac{\text{Total demand}}{\text{Number of days simulated}} = \frac{23}{10} = 2.3$$

Kodak studio can expect an average of 2.3 requests per day for the special colour portraits.

#### **Monte-Carlo Simulation:**

- ✓ Clearly define the problem in order to determine the objectives and constraints.
- ✓ Formulate the appropriate model and decide which variables are to be induced.
- ✓ Specify values of variables to be tested supply values for input parameters and measure the output values.
- ✓ Collect the information required and determine the functional relationship and the types of probability distribution, which are to apply.
- ✓ Define a coding system that will correlate the factors identified in step 1 with the random numbers that will be generated for the simulation.
- ✓ Select a random number generated and create the random numbers to be used in the simulation.
- ✓ Correlate the generated random numbers with the factors identified in step 1 and 5.
- ✓ Summarize and examine the results in an appropriate table.
- ✓ Evaluate the results of the simulation and select best course of action.
- ✓ Formulate proposals for advice to management on new policies and modify the model if necessary.

#### **Illustration:**

The occurrence of rain in a city on a day is dependent upon whether or not it rained on the previous day. If it rained on the previous day, the rain distribution is,

<i>Event</i>	<i>No Rain</i>	<i>1 cm Rain</i>	<i>2 cm Rain</i>	<i>3 cm Rain</i>	<i>4 cm Rain</i>	<i>5 cm Rain</i>
<b>Probability</b>	0.50	0.25	0.15	0.05	0.03	0.02

If it did not rain on the previous day, the rain distribution is

<i>Event</i>	<i>No Rain</i>	<i>1 cm Rain</i>	<i>2 cm Rain</i>	<i>3 cm Rain</i>
<b>Probability</b>	0.75	0.15	0.06	0.04

Simulate the city's weather for 10 days and determine by simulation the total days without rain as well as the total rainfall during the period. Use the following random numbers.

67 63 39 55 29 78 79 06 78 90

for simulation. Assume that for the first day of the simulation it had not rained the day before.

**Solution:** The simulation of the city's weather with and without rainfall in the following tables

#### **Previous Day Rain Distribution:**

<i>Event</i>	<i>Probability</i>	<i>Cumulative Probability</i>	<i>Random Number Range</i>
<i>No Rain</i>	0.50	0.50	00 – 49
<i>1 cm Rain</i>	0.25	0.75	50 – 74
<i>2 cm Rain</i>	0.15	0.90	75 – 89
<i>3 cm Rain</i>	0.05	0.95	90 – 94
<i>4 cm Rain</i>	0.03	0.98	95 – 97
<i>5 cm Rain</i>	0.02	1.00	98 – 99

**Previous Day No Rain Distribution:**

<i>Event</i>	<i>Probability</i>	<i>Cumulative Probability</i>	<i>Random Number Range</i>
<i>No Rain</i>	0.75	0.75	00 – 74
<i>1 cm Rain</i>	0.15	0.90	75 – 89
<i>2 cm Rain</i>	0.06	0.96	90 – 95
<i>3 cm Rain</i>	0.04	1.00	96 – 99

**Simulate for 10 Days Using the Given Random Numbers:**

<i>Day</i>	<i>Random Number</i>	<i>Events</i>	<i>Cumulative Rain</i>
1	67	<i>No Rain</i>	–
2	63	<i>No Rain</i>	–
3	39	<i>No Rain</i>	–
4	55	<i>No Rain</i>	–
5	29	<i>No Rain</i>	–
6	78	<i>1 cm Rain</i>	<i>1 cm</i>
7	79	<i>1 cm Rain</i>	<i>2 cm</i>
8	06	<i>No Rain</i>	<i>2 cm</i>
9	78	<i>1 cm Rain</i>	<i>3 cm</i>
10	90	<i>2 cm Rain</i>	<i>5 cm</i>

During the simulated period it did not rain on 6 out of 10 days. The total rainfall during the period is 5 *cm*.

**Simulation in Business Processing:**

**Simulation in Investment and Budgeting:**

Many business firms invest large sums to expand capacity, reduce production cost and so on. There is considerable risk associated with each investment plan, and this risk can be minimized if more is known about the effects of the many factors by evaluating the alternative courses of action. If these alternatives involve many parameters and interaction with large volume of data, it becomes difficult for the human mind to digest and analyze all the relevant information. Simulation offers a great deal of help by reducing the complexities in such cases.

The readers are no doubt familiar with the evaluation of investment proposals using Discounted Cash Flow Method. The discounted rate of return is based on certain items of information, such as investment, sales volume and prices, costs of production, tax rates, and project life and so on. If there is a variation in one or more of these parameters, the discounted rate of return is bound to change. By varying one or more of these factors, it is possible to arrive at the new discounted rate of return, under each set of varying conditions, and the effect on the rate of return. The simulation method therefore enables the evaluation, without actually incurring the investment and waiting for a long time. It also points out to the decision-maker the importance of one or more parameters over others, on the rate of return.

Budgeting is another area where simulation can be very useful. A simulation model in budgeting simplifies greatly the incorporation of the impact of changes in various variables, which is otherwise manually difficult. A budget consists of many point estimates, like estimate of quantity sold, estimate of selling price, etc. However, it is

preferable to consider a range of probable outcomes, because of the uncertainty attached with various variables. Thus, each point estimate can be substituted by a probability distribution. By the use of simulation, a probability distribution of the expected profits is generated for use in the planning process.

**Illustration:**

An investment corporation wants to study the investment projects based on three factors: market demand in units; price per unit minus cost per unit, and investment required. These factors are felt to be independent of each other. In analyzing a new consumer product, the corporation estimates the following probability distributions.

<i>Annual Demand</i>		<i>(Price – Cost) Per Unit</i>		<i>Investment Required</i>	
<i>Units</i>	<i>Probability</i>	<i>Rs.</i>	<i>Probability</i>	<i>Rs.</i>	<i>Probability</i>
20,000	0.05	3.00	0.10	17,50,000	0.25
25,000	0.10	5.00	0.20	20,00,000	0.50
30,000	0.20	7.00	0.40	25,00,000	0.25
35,000	0.30	9.00	0.20		
40,000	0.20	10.00	0.10		
45,000	0.10				
50,000	0.05				

Using simulation process, repeat the trial 10 times; compute the return on investment for each trial taking these three factors into account. Approximately, what is the most likely return?

**Solution:**

The yearly return can be determined by the formula.

$$\text{Return (R)} = \frac{(\text{Price} - \text{Cost}) \times (\text{Number of Units Demanded})}{\text{Investment}}$$

To determine a cumulative probability distribution corresponding to each of the three factors, we assign an appropriate set of random numbers representing each of the three factors as shown in tables below.

**Market Demand:**

<i>Annual Demand</i>	<i>Probability</i>	<i>Cumulative Probability</i>	<i>Random Numbers</i>
20,000	0.05	0.05	00 – 04
25,000	0.10	0.15	05 – 14
30,000	0.20	0.35	15 – 34
35,000	0.30	0.65	35 – 64
40,000	0.20	0.85	65 – 84
45,000	0.10	0.95	85 – 94
50,000	0.05	1.00	95 – 99

**Price-Cost:**

<i>(Price – Cost) Per Unit</i>	<i>Probability</i>	<i>Cumulative Probability</i>	<i>Random Numbers</i>
3.00	0.10	0.10	00 – 09
5.00	0.20	0.30	10 – 29
7.00	0.40	0.70	30 – 69
9.00	0.20	0.90	70 – 89
10.00	0.10	1.00	90 – 99

**Investment:**

<i>Investment Required</i>	<i>Probability</i>	<i>Cumulative Probability</i>	<i>Random Numbers</i>
17,50,000	0.25	0.25	00 – 24
20,00,000	0.50	0.75	25 – 74
25,00,000	0.25	1.00	75 – 99

We prepare the simulation work-sheet for 10 trials. The simulated return (R) is also calculated by using the formula for R, the results of simulation are shown in table.

**Simulation Worksheet:**



Simulation Run	RN for Demand	Simulated Demand ('000)	RN for Profit	Simulated Profit	RN for Investment	Simulated Investment ('000)	Simulated Return* (%)
1	28	30	19	5.00	18	1750	8.57
2	57	35	07	3.00	67	2000	5.25
3	60	35	90	10.00	16	1750	20.00
4	17	30	02	3.00	71	2000	4.50
5	64	35	57	7.00	43	2000	12.25
6	20	30	28	5.00	68	2000	7.50
7	27	30	29	5.00	47	2000	7.50
8	58	35	83	9.00	24	1750	18.00
9	61	35	58	7.00	19	1750	14.00
10	30	30	41	7.00	97	2500	8.40

$$\text{Simulated Return* (\%)} = \frac{\text{Demand} \times \text{Profit Per Unit}}{\text{Investment}} \times 100$$

The above table shows the highest likely return is 20% which is corresponding to the annual demand of 35,000 units resulting in a profit of Rs. 10 per unit and the required investment will be Rs. 17,50,000.

#### Simulation of an Inventory System:

Many of the inventory problems, especially storage problems, cannot be solved analytically because of the complex nature of the distribution followed by demand or supply. It is, however, possible to get the solution by using simulation techniques. The basic approach would be to determine the probability distribution of the input and output functions from the past data; and run the inventory system artificially by generating the future observations on the assumption of the same distributions. Subsequently, the decision-making regarding the optimization problems would be made by the trial-and-error method.

The artificial samples for future can be generated with the help of random numbers. In inventory control, the reorder point is to be chosen with consideration for the demand during lead time to provide adequate service to customers. If both, the lead time and demand of inventory per unit of time, are random variables, the simulation technique can be used to investigate the effect of different inventory policies.

#### Illustration:

The automobile company manufactures around 150 scooters. The daily production varies from 146 to 154 depending upon the availability of raw materials and other working conditions.

Production (Per Day)	146	147	148	149	150	151	152	153	154
Probability	0.04	0.09	0.12	0.14	0.11	0.10	0.20	0.12	0.08

The finished scooters are transported in a specially arranged lorry accommodating 150 scooters. Using following random numbers,

80 81 76 75 64 43 18 26 10 12 65 68 69 61 57

Simulate the process to find out

(i) What will be the average number of scooters waiting in the factory?

(ii) What will be the average number of empty space on the lorry?

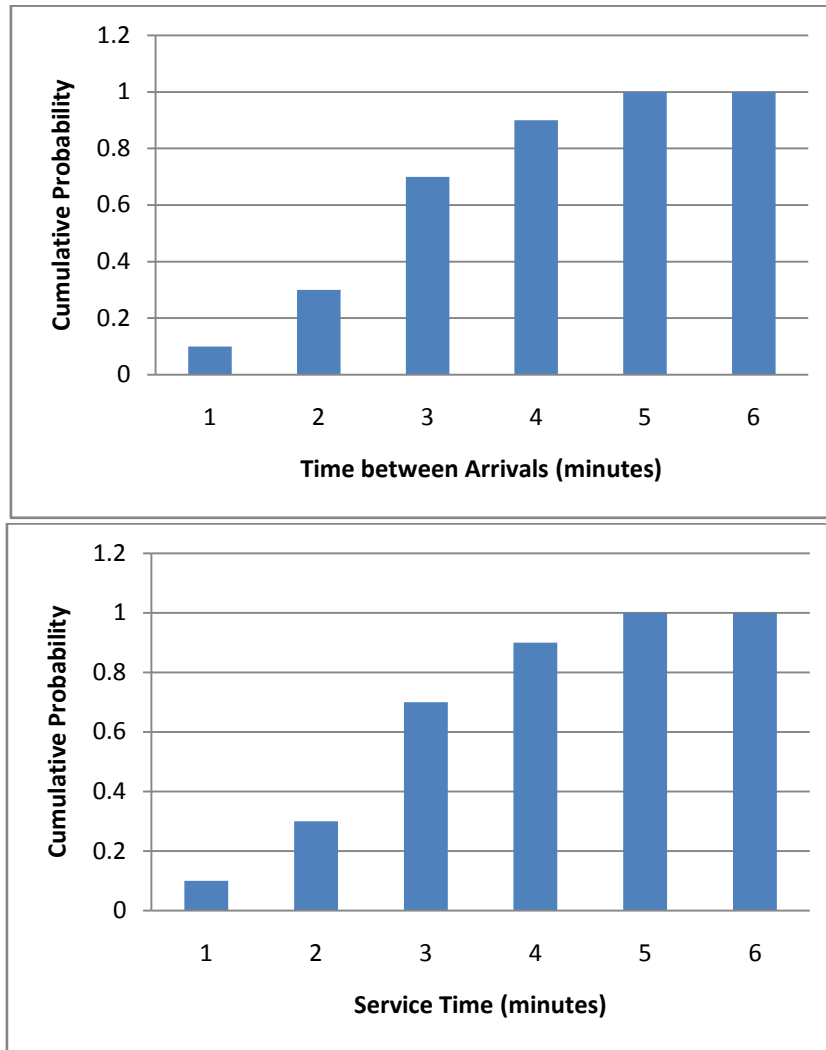
#### Random Number Coding:

Production Per Day	Probability	Cumulative Probability	Random Number Assigned
146	0.04	0.04	00 – 03
147	0.09	0.13	04 – 12
148	0.12	0.25	13 – 24
149	0.14	0.39	25 – 38





From the given frequency distribution of arrivals and service times, the probabilities and cumulative probabilities are first worked out as shown in the following table. These, then become the basis for generating arrival and service times in conjunction with a table a table of random numbers.



#### Cumulative Probabilities:

<i>Time Between Arrivals (Minutes)</i>	<i>Cumulative Probability</i>	<i>Service Time (Minutes)</i>	<i>Cumulative Probability</i>
1 – 2	0.05	1 – 2	0.10
2 – 3	0.25	2 – 3	0.30
3 – 4	0.60	3 – 4	0.70
4 – 5	0.85	4 – 5	0.90
5 – 6	0.95	5 – 6	1.00
6 – 7	1.00	6 – 7	1.00

As we have to use the random number table, first of all we allot the random numbers to various intervals as shown in the table below:

<i>InterArrival Time (Minutes)</i>	<i>Probability</i>	<i>RN Allotted</i>	<i>Service Time (Minutes)</i>	<i>Probability</i>	<i>RN Allotted</i>
1 – 2	0.05	00 – 04	1 – 2	0.10	00 – 09
2 – 3	0.20	05 – 24	2 – 3	0.20	10 – 29
3 – 4	0.35	25 – 59	3 – 4	0.40	30 – 69
4 – 5	0.10	60 – 84	4 – 5	0.20	70 – 89
5 – 6	0.10	85 – 94	5 – 6	0.10	90 – 99
6 – 7	0.05	95 – 99	6 – 7	–	–

**A Simulation Work Sheet has been Developed in the Following Manner:**

The random numbers developed above are related to the cumulative probability distributions of arrival and service times. The first random number of arrival time is 64. This number lies between 60 – 84 and indicates a simulated arrival time of 4 minutes. All simulated arrival and service times have been worked out in a similar fashion.

After generating the arrival and service times from a table of random numbers, the next step is to list the arrival time in the appropriate column. The first arrival comes in 4 minutes after the starting time. This means that the clerk waited for 4 minutes initially. It has been shown under the column-waiting time; clerk. The simulated service for the first arrival is 3 minutes which results in the service being completed by 11.07 am. The next arrival comes at 11.08 am. Which indicates that no one has waited in the queue.

The second arrival comes at 11.05 am. But the services will begin only at 11.07 am. Since the service of first arrival ends at 11.07 am. This means that the second arrival has waited for 2 minutes before the start of its service one customer waiting in the queue is shown in the last column of the simulation table. The procedure has been followed throughout the preparation of the simulation work sheet.

**Simulation Work Sheet:**

Random Number	Inter Arrival Time	Arrival Time (am)	Service Begins (am)	Random Number	Service		Waiting Time		Line Length
					Time (min)	Ends (am)	Clerk	Customer	
64	4	11.04	11.04	30	3	11.07	4	—	—
04	1	11.05	11.07	75	4	11.11	—	2	1
02	1	11.06	11.11	38	3	11.14	—	5	1
70	4	11.10	11.14	24	2	11.16	—	4	1
03	1	11.11	11.16	57	3	11.19	—	5	1
60	4	11.15	11.19	09	1	11.20	—	4	1
16	2	11.17	11.20	12	2	11.22	—	3	1
08	2	11.19	11.22	18	2	11.24	—	3	1
36	3	11.22	11.24	65	3	11.27	—	2	1
38	3	11.25	11.27	25	2	11.29	—	2	1
07	2	11.27	11.29	11	2	11.31	—	2	1
08	2	11.29	11.31	79	4	11.35	—	2	1
59	3	11.32	11.35	61	3	11.38	—	3	1
53	3	11.35	11.38	77	4	11.42	—	3	1
03	1	11.36	11.42	10	2	11.44	—	6	1
62	4	11.40	11.44	16	2	11.45	—	4	1
36	3	11.43	11.46	55	3	11.49	—	3	1
27	3	11.46	11.49	52	3	11.52	—	3	1
97	6	11.52	11.52	59	3	11.55	—	—	—
86	5	11.57	11.57	63	3	12.00	2	—	—
<b>20</b>	<b>57</b>				<b>54</b>		<b>6</b>	<b>56</b>	<b>17</b>

The following information can be obtained from the simulation work sheet based on the period of one hour only;

$$(a) \text{ Average queue length} = \frac{\text{Number of Customers in the Waiting line}}{\text{Number}} = \frac{17}{20} = 0.85$$

$$(b) \text{ Average Waiting time of Customer before service} = \frac{\text{Customer Waiting time}}{\text{Number of Arrivals}} = \frac{56}{20} = 2.80 \text{ minutes}$$

$$(c) \text{ Average Service time} = \frac{\text{Total Service time}}{\text{Number of Arrivals}} = \frac{54}{20} = 2.70 \text{ minutes}$$

$$(d) \text{ Time a Customer Spends in the System} = \text{Average Service time} + \text{Average Waiting time before service} = 2.70 + 2.80 = 5.50 \text{ minutes}$$

Simulation work sheet developed in the problem also states that if one or more clerk is added, there is no need for a customer to wait in the queue. But before effecting any decision, the cost of having an additional clerk has to be compared with the cost due to customer waiting time. This can be worked out as follows

<i>One Hour Period</i>	<i>Cost with One Clerk</i>	<i>Cost with Two Clerks</i>
<i>Customer waiting time</i> (56 minutes $\times$ Rs. 5 per Hour)	Rs. 4.50	Nil
<i>Clerk's Cost</i>	Rs. 6.00	Rs. 12.00
<i>Total cost of</i> <i>One Hour Period</i>	Rs. 10.50	Rs. 12.00

If the above analysis based on simulation for a period of one hour only is representative of the actual situation, then it may be concluded that the cost with one clerk is lower than what it is with two clerks. Hence, it would not be an economical proposition to engage an additional clerk.

#### **Conclusion:**

Simulation is indeed a versatile tool. It provides one statistical estimates rather than exact results and it only compares the alternatives rather than generating an optimal one. It is a slow and costly way to study a problem. Despite limitations, it is an invaluable tool in Operations Research.

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